TRANSFORMATIONS OF CONJUGATE SYSTEMS WITH EQUAL POINT INVARIANTS*

BY

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In a recent memoir† we have established a transformation of a surface S into a surface S_1 such that the lines joining corresponding points form a congruence whose developables meet S and S_1 in conjugate systems with equal tangential invariants; we have called it a *transformation* Ω. In the present paper we consider a transformation of a surface S into a surface S_1 such that the developables of the congruence of joins of corresponding points meet S and S_1 in conjugate systems with equal point invariants. In both cases the fundamental operation is the transformation of Moutard‡ of an equation of the Laplace type with equal invariants, just as this transformation lies at the base of the various types of transformations in which the two surfaces are the focal sheets of a W-congruence. We identify the transformation under discussion with one which is involved in part in a theorem established from a different point of view by Koenigs, who at the time was considering an entirely different problem.§ We make a detailed study of this transformation,‖ because it seems to be the foundation of certain important transformations which have been established in other ways and it points the way to other important transformations. The transformations K admit a theorem of permutability just as the other two general types of transformations referred to above. So far as we know every class of transformations of surfaces of a particular kind into surfaces of the same kind is reducible to one of these three types. Consequently we not only know the basis for the existence of a theorem of permutability in the known particular transformations, but in subsequent investigations we shall have a more immediate method of establishing such a theorem for transformations under consideration. An example of this is given at the close of the memoir.

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† *Conjugate systems with equal tangential invariants and the transformation of Moutard*, Rendiconti del circolo matematico di Palermo, vol. 38.

‡ Journal de l'école polytechnique, cahier 45 (1878), p. 1; cf. also, Darboux, *Leçons*, vol. 2, p. 145.

§ Comptes Rendus, vol. 113 (1891), p. 1022.

‖ Referred to in the following pages as a transformation K.

We show that transformations K are commutative with certain *radial transformations*, under which the points of a surface are transformed along lines which are concurrent. These transformations enable us to deduce the general theorem of permutability for transformations K from a special case in which the four surfaces consist of two pairs of associate surfaces, that is, two surfaces corresponding with parallelism of tangent planes and such that to asymptotic lines on either correspond a conjugate system on the other.

Although the treatment of §§ 1–5 is stated in terms of ordinary space, the results are equally true in n-space, if we substitute for a surface referred to a conjugate system a *reseau*, as defined by Guichard* and use the term *congruence* in the restricted sense given to it by this author.

The remainder of the memoir deals with ordinary space. It is shown that each pair of surfaces in the relation of a transformation K gives rise to four surfaces Σ, Σ_1, Σ_2, Σ', such that the pairs of surfaces $\Sigma\Sigma_1$, $\Sigma\Sigma_2$, $\Sigma_1\Sigma'$, $\Sigma_2\Sigma'$ are the focal surfaces of W-congruences, and conversely, when four surfaces are so related there exist four pairs of surfaces each pair of which are in the relation of a transformation K.

In order to apply the transformations K to particular types of surfaces it is desirable to put the equations in the form developed in § 9. The memoir contains one application, namely the solution of the problem of determining when S and S_1 envelop a two-parameter family of spheres. It is shown that in this case S and S_1 are isothermic surfaces in the relation of the transformation discovered by Darboux† and studied at length by Bianchi.‡ The theorem of permutability of these transformations is an immediate consequence of our general theorem.

The closing section of the memoir contains the solution of the problem under what conditions does the congruence of joins of corresponding points on two surfaces in the relation of a transformation K possess other pairs of points which describe surfaces similarly related. It is shown that, if a congruence possesses more than one pair of such points, it possesses an infinity of pairs.

In a subsequent memoir we shall consider the case where the joins of corresponding points on two surfaces S and S_1 form a normal congruence and we shall show that the surfaces orthogonal to such a congruence admit a transformation of the Ribaucour type.

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*Annales de l'école normale supérieure, sér. 3, vol. 14 (1897), pp. 467–516.

†Annales de l'école normale supérieure, sér. 3, vol. 16 (1899), pp. 491–508.

‡Annali di Matematica, ser. 3, vol. 11 (1905), pp. 93–158.

1. Fundamental transformation

By definition if x, y, z are the cartesian coördinates of a surface S referred to a conjugate system with equal point invariants, these functions are solutions of an equation of the form

$$(1)\qquad \frac{\partial^2 \theta}{\partial u\, \partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v} \frac{\partial \theta}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u} \frac{\partial \theta}{\partial v} = 0,$$

where ρ is in general a function of u and v.

In accordance with the transformation of Moutard*, if θ is any solution of equation (1), the function ξ defined by the equations

$$(2)\qquad \frac{\partial \xi}{\partial u} = -\rho\left(x\frac{\partial \theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right), \qquad \frac{\partial \xi}{\partial v} = \rho\left(x\frac{\partial \theta}{\partial v} - \theta\frac{\partial x}{\partial v}\right)$$

is the solution of another equation with equal invariants. In fact, if we differentiate the first of (2) with respect to v, we find readily that ξ is a solution of the equation

$$(3)\qquad \frac{\partial^2 \theta'}{\partial u\, \partial v} - \frac{\partial \log \sqrt{\rho}\, \theta}{\partial v} \frac{\partial \theta'}{\partial u} - \frac{\partial \log \sqrt{\rho}\, \theta}{\partial u} \frac{\partial \theta'}{\partial v} = 0.$$

If in (2) we replace ξ and x by the quantities η and y respectively, and also by ζ and z, the functions η and ζ so defined satisfy equation (3).

When the functions ξ, η, ζ are replaced by λx_1, λy_1, λz_1, the necessary and sufficient condition that the surface S_1, whose coördinates are x_1, y_1, z_1, is referred to a conjugate system is that λ be a solution of equation (3). We assume that λ is such a function and write equations (2) in the form

$$(4)\qquad \frac{\partial}{\partial u}(\lambda x_1) = -\rho\left(x\frac{\partial \theta}{\partial u} - \theta\frac{\partial x}{\partial u}\right), \qquad \frac{\partial}{\partial v}(\lambda x_1) = \rho\left(x\frac{\partial \theta}{\partial v} - \theta\frac{\partial x}{\partial v}\right).$$

It is readily shown that x_1, y_1, z_1 are solutions of

$$(5)\qquad \frac{\partial^2 \phi}{\partial u\, \partial v} + \frac{\partial}{\partial v}\log\frac{\lambda}{\sqrt{\rho}\,\theta}\cdot\frac{\partial \phi}{\partial u} + \frac{\partial}{\partial u}\log\frac{\lambda}{\sqrt{\rho}\,\theta}\cdot\frac{\partial \phi}{\partial v} = 0.$$

Hence the parametric conjugate system on S_1 has equal point invariants.

We denote by M and M_1 corresponding points on S and S_1 and by x_0, y_0, z_0 the coördinates of a point M_0 on the line joining M and M_1. Evidently

$$(6)\qquad x_0 = x + t(x_1 - x), \qquad y_0 = y + t(y_1 - y), \qquad z_0 = z + t(z_1 - z),$$

where t is a function of u and v.

By means of equations (4) the first derivatives of x_0 assume the form

* Loc. cit.

$$
(7)\qquad
\begin{aligned}
\frac{\partial x_0}{\partial u} &= \left[t\left(\frac{\rho\theta}{\lambda}-1\right)+1\right]\frac{\partial x}{\partial u}+(x_1-x)\left(\frac{\partial t}{\partial u}-t\frac{\partial \log\lambda}{\partial u}\right)\\
&\qquad -\frac{xt}{\lambda}\left(\frac{\partial\lambda}{\partial u}+\rho\frac{\partial\theta}{\partial u}\right),\\
\frac{\partial x_0}{\partial v} &= \left[1-t\left(1+\frac{\rho\theta}{\lambda}\right)\right]\frac{\partial x}{\partial v}+(x_1-x)\left(\frac{\partial t}{\partial v}-t\frac{\partial \log\lambda}{\partial v}\right)\\
&\qquad -\frac{xt}{\lambda}\left(\frac{\partial\lambda}{\partial v}-\rho\frac{\partial\theta}{\partial v}\right).
\end{aligned}
$$

It is our purpose to show that λ can be so chosen that the developables of the congruence (G) of lines joining corresponding points M and M_1 meet the surfaces S and S_1 in the parametric conjugate systems with equal point invariants.

In the first place we observe that 1 is a solution of equation (1) and consequently it follows from (2), on replacing x by 1, that the equations

$$
(8)\qquad \partial\lambda/\partial u = -\rho\,\partial\theta/\partial u, \qquad \partial\lambda/\partial v = \rho\,\partial\theta/\partial v
$$

are consistent and that the function λ so defined is a solution of equation (3).

If we substitute this value of λ in equations (7) and give to t either of the values

$$
(9)\qquad t_1 = \lambda/(\lambda-\rho\theta), \qquad t_2 = \lambda/(\lambda+\rho\theta),
$$

we obtain respectively

$$
\frac{\partial x_0/\partial u}{x_1-x}=\frac{\partial y_0/\partial u}{y_1-y}=\frac{\partial z_0/\partial u}{z_1-z}, \qquad \frac{\partial x_0/\partial v}{x_1-x}=\frac{\partial y_0/\partial v}{y_1-y}=\frac{\partial z_0/\partial v}{z_1-z}.
$$

Hence the parametric ruled surfaces of the congruence (G) are developable.

Furthermore, if F_1 and F_2 denote the focal points whose coördinates are given by (6) when t is replaced by t_1 and t_2 from (9), it is readily shown that M and M_1 are harmonic to F_1 and F_2.

For the function λ given by equations (8) the equations (4) assume the form

$$
(10)\qquad \frac{\partial x_1}{\partial u}=\frac{\rho}{\lambda}\left[(x_1-x)\frac{\partial\theta}{\partial u}+\theta\frac{\partial x}{\partial u}\right], \qquad \frac{\partial x_1}{\partial v}=-\frac{\rho}{\lambda}\left[(x_1-x)\frac{\partial\theta}{\partial v}+\theta\frac{\partial x}{\partial v}\right].
$$

The foregoing results are stated in

THEOREM 1. *When a surface S referred to a conjugate system with equal point invariants is known, and also any solution θ of the point equation of S, the coördinates of a surface S_1 can be found by quadratures* (10), *and this surface S_1 is such that the developables of the congruence of lines joining corresponding points on S and S_1 meet these surfaces in conjugate systems with equal point invariants. Moreover, the focal points of the congruence are harmonic to the corresponding points on S and S_1.*

From (10) it follows that the necessary and sufficient condition that the tangents to the parametric curves on S and S_1 at corresponding points be parallel is that θ be constant. From equations (8) it follows that λ also must be constant. If we take $\theta = -\lambda = 1$, equations (10) become

$$\frac{\partial x_1}{\partial u} = -\rho\frac{\partial x}{\partial u}, \qquad \frac{\partial x_1}{\partial v} = \rho\frac{\partial x}{\partial v}. \tag{11}$$

These equations show that in this case S and S_1 are associate surfaces.* Moreover, it is known that the corresponding conjugate systems on any two associate surfaces have equal point invariants and their equations are of the form (11). Hence the problem of associate surfaces is a special case of the general problem which we are considering.

2. The theorem of Koenigs. The transformations K

In this section we show that by means of equations (10) we obtain the most general surface S_1 such that the congruence of lines joining corresponding points of S and S_1 possess the properties indicated in Theorem 1. We begin by looking upon the congruence (G) as composed of the tangents to the curves $v =$ const. on a surface S_0, whose coördinates are x_0, y_0, z_0 and on which the parametric system is conjugate.

Lucien Levy† has shown that the cartesian coördinates x, y, z of any point M of a line of the congruence (G) which describes a surface S cut in a conjugate system by the developables of (G) are expressible in the form

$$x = x_0 - \frac{\sigma}{\frac{\partial \sigma}{\partial u}}\frac{\partial x_0}{\partial u}, \qquad y = y_0 - \frac{\sigma}{\frac{\partial \sigma}{\partial u}}\frac{\partial y_0}{\partial u}, \qquad z = z_0 - \frac{\sigma}{\frac{\partial \sigma}{\partial u}}\frac{\partial z_0}{\partial u}, \tag{12}$$

where σ is a solution of the point equation of S_0, namely

$$\frac{\partial^2 \sigma}{\partial u\,\partial v} + a\frac{\partial \sigma}{\partial u} + b\frac{\partial \sigma}{\partial v} = 0. \tag{13}$$

Moreover, each solution of this equation leads to such a surface S.

We consider a second surface S_1 with coördinates x_1, y_1, z_1 given by

$$x_1 = x_0 - \frac{\sigma_1}{\frac{\partial \sigma_1}{\partial u}}\frac{\partial x_0}{\partial u}, \qquad y_1 = y_0 - \frac{\sigma_1}{\frac{\partial \sigma_1}{\partial u}}\frac{\partial y_0}{\partial u}, \qquad z_1 = z_0 - \frac{\sigma_1}{\frac{\partial \sigma_1}{\partial u}}\frac{\partial z_0}{\partial u}, \tag{14}$$

σ_1 being another solution of (13).

* E. p. 380. A reference of this kind is to the author's *Differential Geometry*, Ginn and Company, Boston (1909).

† *Sur quelques équations linéaires aux derivées partielles*, Journal de l'école polytechnique, cahier 56 (1886), p. 77.

For the sake of brevity we put

$$
(14') \qquad \begin{aligned} M &= a + \frac{1}{\sigma}\frac{\partial \sigma}{\partial v}, \qquad & N &= b + \frac{1}{\sigma}\frac{\partial \sigma}{\partial u}, \\ h &= \frac{\partial a}{\partial u} + ab, & k &= \frac{\partial b}{\partial v} + ab. \end{aligned}
$$

Thus h and k are the invariants of equation (13). It follows from (13) and (14′) that

$$
(15) \qquad \frac{\partial M}{\partial u} = h - MN, \qquad \frac{\partial N}{\partial v} = k - MN.
$$

In accordance with the theory of the Laplace transformation* of equation (13) the coördinates of the other focal point of the congruence (G) are of the form

$$
x_0' = x_0 + \frac{1}{b}\frac{\partial x_0}{\partial u}.
$$

The condition that the focal points be harmonic with respect to the corresponding points on S and S_1, whose coördinates are given by (12) and (14) is

$$
(16) \qquad N_1 + N = 0,
$$

where N_1 is given by (14′) when σ is replaced by σ_1. If we express the condition that N_1 satisfies an equation analogous to the second of (15), we have

$$
(17) \qquad M_1 = M - 2k/N,
$$

where M_1 is given by (14′) when σ is replaced by σ_1. If we require that M_1 satisfies an equation analogous to the first of (15) we get

$$
(18) \qquad \frac{\partial N}{\partial u} = N\frac{\partial \log k}{\partial u} + \frac{N^2}{k}(MN - k).
$$

Whenever equations (13) and (18) admit a common solution σ, this function and σ_1, which follows by quadratures from (16) and (17), determine two surfaces S and S_1 in the relation of a transformation K.

Later (§ 11) we shall consider the conditions under which equations (13) and (18) possess a common solution. Now we are interested in showing that when these conditions are satisfied the coördinates of the two surfaces are in the relations expressed by equations (10).

From (12) we obtain by differentiation

$$
(19) \qquad \begin{aligned} \frac{\partial x}{\partial u} &= \sigma\left(\frac{\partial \sigma}{\partial u}\right)^{-2}\left(\frac{\partial^2 \sigma}{\partial u^2}\frac{\partial x_0}{\partial u} - \frac{\partial \sigma}{\partial u}\frac{\partial x_0}{\partial v}\right), \\ \frac{\partial x}{\partial v} &= \sigma\left(\frac{\partial \sigma}{\partial u}\right)^{-2} N\left(-\frac{\partial \sigma}{\partial v}\frac{\partial x_0}{\partial u} + \frac{\partial \sigma}{\partial u}\frac{\partial x_0}{\partial v}\right). \end{aligned}
$$

* E. p. 404.

The expression for $\partial^2 x/\partial u\, \partial v$ obtained from either of these equations can be put in the form

$$(20)\qquad \frac{\partial^2 x}{\partial u\, \partial v} = \left(\frac{\partial \sigma}{\partial u}\right)^{-1}\left(N\frac{\partial \sigma}{\partial v}\frac{\partial x}{\partial u} + \frac{\dfrac{\partial \sigma}{\partial u}\left(\dfrac{\partial b}{\partial u} - b^2\right) - b\dfrac{\partial^2 \sigma}{\partial u^2}}{N}\frac{\partial x}{\partial v}\right).$$

A necessary and sufficient condition that this equation be of the form (1) is that there exist a function ρ defined by

$$(21)\qquad \begin{aligned} \frac{\partial \log \sqrt{\rho}}{\partial u} &= -\left(\frac{\partial \sigma}{\partial u}\right)^{-1} N\frac{\partial \sigma}{\partial v}, \\ \frac{\partial \log \sqrt{\rho}}{\partial v} &= -\frac{\dfrac{\partial b}{\partial u} - b^2 - b\left(\dfrac{\partial \sigma}{\partial u}\right)^{-1}\dfrac{\partial^2 \sigma}{\partial u^2}}{N}. \end{aligned}$$

It is readily shown that these equations are consistent if equation (18) is satisfied. Owing to the fact that the relation between S and S_1 is reciprocal, it follows that the parametric system on S_1 also is conjugate with equal point invariants. Hence we have the theorem of Koenigs:*

THEOREM 2. *If two surfaces S and S_1 are so related that the focal points of the congruence of lines joining corresponding points M and M_1 on these surfaces are harmonic with respect to these points, and also if the developables of the congruence meet S and S_1 in conjugate systems, the latter have equal point invariants.*

Now we show that coördinates the of S and S_1 are in the relation of equations (10).

Since σ_1 determined by quadratures from (16) and (17) is a solution of equation (13), it is evident from (12) that the function θ given by

$$(22)\qquad \theta = \sigma_1 - \frac{\sigma}{\dfrac{\partial \sigma}{\partial u}}\frac{\partial \sigma_1}{\partial u} = 2\sigma_1\left(1 + b\frac{\sigma}{\dfrac{\partial \sigma}{\partial u}}\right)$$

is a solution of equation (20).

The first derivatives of θ are given by equations obtained from equations (19) on replacing x_0 by σ_1; and from these same equations on substituting σ_1 for σ we get the first derivatives of x_1. When these values are substituted in equations (10), they are satisfied, provided that

$$(23)\qquad \lambda = \rho\left(\sigma_1 - \frac{\sigma}{\dfrac{\partial \sigma}{\partial u}}\frac{\partial \sigma_1}{\partial u}\right)\frac{\sigma}{\dfrac{\partial \sigma}{\partial u}}\frac{\partial \sigma_1}{\partial u}\Big/\sigma_1 = \rho\frac{\theta}{\sigma_1}(\sigma_1 - \theta).$$

When this expression for λ is substituted in equations (8), we are led to equations (21). Hence we have

* Loc. cit.

THEOREM 3. *The transformation* (10) *of a surface S into a surface S_1 is the most general case of pairs of surfaces in the relation of Koenigs.*

In view of this result we shall hereafter say that S_1 given by (10) is obtained from S by a *transformation* K.

3. THEOREM OF PERMUTABILITY

Suppose that we have two solutions θ_1 and θ_2 of equation (1) and consider the two surfaces S_1 and S_2 which arise from S by transformations K_1 and K_2 determined by θ_1 and θ_2 respectively. The coördinates x_1, y_1, z_1 and x_2, y_2, z_2 are given by equations of the form (10), namely

$$(24)\qquad \begin{aligned} \frac{\partial x_i}{\partial u} &= \frac{\rho}{\lambda_i}\left[(x_i - x)\frac{\partial \theta_i}{\partial u} + \theta_i \frac{\partial x}{\partial u}\right], \\ \frac{\partial x_i}{\partial v} &= -\frac{\rho}{\lambda_i}\left[(x_i - x)\frac{\partial \theta_i}{\partial v} + \theta_i \frac{\partial x}{\partial v}\right] \end{aligned} \qquad (i = 1, 2),$$

where, in accordance with (8),

$$(25)\qquad \frac{\partial \lambda_i}{\partial u} = -\rho\frac{\partial \theta_i}{\partial u}, \qquad \frac{\partial \lambda_i}{\partial v} = \rho\frac{\partial \theta_i}{\partial v} \qquad (i = 1, 2).$$

From (5) it follows that the functions x_i, y_i, z_i, defined by (24) and similar equations in y_i and z_i, satisfy the corresponding equation

$$(26)\qquad \frac{\partial^2 \phi_i}{\partial u\, \partial v} + \frac{\partial}{\partial v}\log\frac{\lambda_i}{\sqrt{\rho}\,\theta_i}\frac{\partial \phi_i}{\partial u} + \frac{\partial}{\partial u}\log\frac{\lambda_i}{\sqrt{\rho}\,\theta_i}\frac{\partial \phi_i}{\partial v} = 0 \qquad (i = 1, 2).$$

It is evident from (4) that these equations are satisfied also by the functions θ_1' and θ_2' respectively, defined by

$$(27)\qquad \begin{aligned} \frac{\partial}{\partial u}(\theta_i'\lambda_i) &= -\rho\left(\theta_i\frac{\partial \theta_j}{\partial u} - \theta_j\frac{\partial \theta_i}{\partial u}\right), \\ \frac{\partial}{\partial v}(\theta_i'\lambda_i) &= \rho\left(\theta_i\frac{\partial \theta_j}{\partial v} - \theta_j\frac{\partial \theta_i}{\partial v}\right) \end{aligned} \qquad \left(\begin{matrix} i = 1, 2, \\ j = 1, 2; \end{matrix}\ i \neq j\right).$$

As thus defined each of the functions θ_i' is determined only to within the additive function c_i/λ_i, where c_i is an arbitrary constant. Hereafter in using pairs of functions θ_1' and θ_2' simultaneously we understand that the constants c_i are so determined that

$$(28)\qquad \theta_1'\lambda_1 + \theta_2'\lambda_2 = 0,$$

which evidently is possible because of the form of (27).

It is our purpose to show that, if we use the functions θ_1' and θ_2' to determine transformations K_1' and K_2' of S_1 and S_2 respectively, the resulting surfaces coincide, giving a surface S' with coördinates x', y', z'.

In the first place we observe that if we put

$$\rho_i = \lambda_i^2/\rho\theta_i^2, \tag{29}$$

equations (26) may be given a form similar to (1). Hence for the transformations K_1' and K_2' the equations analogous to (25) are

$$\frac{\partial \lambda_i'}{\partial u} = -\frac{\lambda_i^2}{\rho\theta_i^2}\frac{\partial \theta_i'}{\partial u}, \qquad \frac{\partial \lambda_i'}{\partial v} = \frac{\lambda_i^2}{\rho\theta_i^2}\frac{\partial \theta_i'}{\partial v} \qquad (i = 1, 2). \tag{30}$$

In order that the transforms of S_1 and S_2 may coincide, equations analogous to (24) necessitate the conditions

$$\begin{aligned} \frac{\partial x'}{\partial u} &= \frac{\rho_1}{\lambda_1'}\left[(x' - x_1)\frac{\partial \theta_1'}{\partial u} + \theta_1'\frac{\partial x_1}{\partial u}\right] = \frac{\rho_2}{\lambda_2'}\left[(x' - x_2)\frac{\partial \theta_2'}{\partial u} + \theta_2'\frac{\partial x_2}{\partial u}\right], \\ \frac{\partial x'}{\partial v} &= -\frac{\rho_1}{\lambda_1'}\left[(x' - x_1)\frac{\partial \theta_1'}{\partial v} + \theta_1'\frac{\partial x_1}{\partial v}\right] = -\frac{\rho_2}{\lambda_2'}\left[(x' - x_2)\frac{\partial \theta_2'}{\partial v} + \theta_2'\frac{\partial x_2'}{\partial v}\right]. \end{aligned} \tag{31}$$

If we substitute in the second equation of the first row the expressions for $\partial\theta_1'/\partial u$ and $\partial x_i/\partial u$ given by (24) and (27), we obtain

$$\left(\theta_2\frac{\partial \theta_1}{\partial u} - \theta_1\frac{\partial \theta_2}{\partial u}\right)P + \frac{\lambda_1\theta_1'}{\theta_2}\left(\frac{1}{\lambda_1'\theta_1} + \frac{1}{\lambda_2'\theta_2}\right)\left[(x' - x)\frac{\partial \theta_2}{\partial u} + \theta_2\frac{\partial x}{\partial u}\right] = 0, \tag{32}$$

where for the sake of brevity we have put

$$P = x'\left[\frac{\lambda_1}{\lambda_1'\theta_1^2\theta_2}(\theta_1' + \theta_2) + \frac{\lambda_2}{\lambda_2'\theta_2'}\right] - \left(\frac{x_1\lambda_1}{\lambda_1'\theta_1^2} + \frac{x_2\lambda_2}{\lambda_2'\theta_2^2}\right) - \frac{\lambda_1\theta_1' x}{\theta_1^2\theta_2\lambda_1'}. \tag{33}$$

In a similar manner we obtain from the second row of (31) the equation

$$\left(\theta_2\frac{\partial \theta_1}{\partial v} - \theta_1\frac{\partial \theta_2}{\partial v}\right)P + \frac{\lambda_1\theta_1'}{\theta_2}\left(\frac{1}{\lambda_1'\theta_1} + \frac{1}{\lambda_2'\theta_2}\right)\left[(x' - x)\frac{\partial \theta_2}{\partial v} + \theta_2\frac{\partial x}{\partial v}\right] = 0. \tag{34}$$

The equations of condition (32) and (34) are satisfied if $P = 0$ and if

$$\lambda_1'\theta_1 + \lambda_2'\theta_2 = 0. \tag{35}$$

We shall show that these conditions are satisfied and lead to the desired result.

If we differentiate equation (35) with respect to u and v separately and require that the functions λ_i' shall satisfy (30), we find that we must have

$$\lambda_1'\theta_1 = \lambda_1(\theta_1' + \theta_2) - \lambda_2\theta_1, \qquad \lambda_2'\theta_2 = \lambda_2(\theta_2' + \theta_1) - \lambda_1\theta_2. \tag{36}$$

In consequence of (28) these equations are consistent with (35) and furthermore these expressions for λ_i' satisfy equations (30).

Because of (36) the equation obtained by equating P to zero is reducible to

$$\lambda_1'\theta_1 x' = \theta_1'\lambda_1 x + \lambda_1\theta_2 x_1 - \lambda_2\theta_1 x_2, \tag{37}$$

which, in consequence of (28) and (35), is equivalent to

$$(38)\qquad \lambda_2' \, \theta_2 \, x' = \theta_2' \, \lambda_2 \, x + \lambda_2 \, \theta_1 \, x_2 - \lambda_1 \, \theta_2 \, x_1 .$$

With the aid of equations (36) it is shown that the value of x' given by (37) satisfies equations (31). Hence not only does S' exist satisfying the above requirements, but its coördinates can be found without quadrature after θ_1' and θ_2' have been obtained by the quadratures (27).

We shall say that four surfaces S, S_1, S_2, S' in the above relation form a *quatern*. From (37) it is evident that the corresponding points on the four surfaces lie in a plane.

From the form of (27) it is seen that θ_1' is determined to within the additive quantity c/λ_1, where c is an arbitrary constant. The first of equations (36) shows that the corresponding change in λ_1' is the addition of c/θ_1. Hence if θ_1' is a function giving a surface S' and if we denote by S_c' the surface determined by $\theta_1' + c/\lambda_1$, the coördinates x_c', y_c', z_c' of S_c' are expressible in the form

$$(\lambda_1' \, \theta_1 + c) \, x_c' = (\theta_1' \, \lambda_1 + c) \, x + \lambda_1 \, \theta_2 \, x_1 - \lambda_2 \, \theta_1 \, x_2 ,$$

which may be written also

$$(39)\qquad (\lambda_1' \, \theta_1 + c) \, x_c' = cx + \lambda_1' \, \theta_1 \, x' .$$

Hence all the corresponding points on the family of surfaces S_c' lie on the line joining M and M', and each point divides the segment MM' in the ratio $c/\lambda_1' \, \theta_1$.

The foregoing results may be assembled into

THEOREM 4. *If S_1 and S_2 are two surfaces arising from S by transformations K, there can be found by a quadrature an infinity of surfaces S' each of which is in the relation of transformations K with both S_1 and S_2. The corresponding points on the suite of surfaces S' lie on a line which passes through the corresponding point on S and in the plane determined by the latter point and the corresponding points on S_1 and S_2.*

We consider in particular the case when S_1 is an associate surface of S. As shown in § 1 this arises when we take $\theta_1 = -\lambda_1 = 1$. Now equations (27) for $i = 1$ become

$$\frac{\partial \theta_1'}{\partial u} = \rho \frac{\partial \theta_2}{\partial u}, \qquad \frac{\partial \theta_1'}{\partial v} = -\rho \frac{\partial \theta_2}{\partial v}.$$

Comparing these with (25) and making use of (28) we see that

$$\theta_1' = -\lambda_2 + c, \qquad \theta_2' = -1 + c/\lambda_2 .$$

Hence if we take $c = 0$ the surfaces S_2 and S' are associate. In this case

equation (37) can be written

$$\theta_2 (x' - x_1) = \lambda_2 (x_2 - x). \tag{40}$$

There ore we have

THEOREM 5. *If S and S_1 are two associate surfaces, and if S_2 is any surface arising from S by a transformation K, there exists a surface S' associate to S_2 which is in the relation of a transformation K with S_1; moreover, the lines joining corresponding points of S_1 and S' and of S and S_2 are parallel.*

4. ENVELOPE OF THE PLANES OF A QUATERN

We have just seen that the four corresponding points M, M_1, M_2, M' of the surfaces of a quatern S, S_1, S_2, S' lie in a plane. Since this plane contains the lines MM_1 and MM_2 which generate congruences whose developables meet S in a conjugate system, it envelopes a surface upon which the curves corresponding to these developables form a conjugate system, as follows from the general theory of conjugate systems and congruences.* Moreover, if Π is the point of the envelope corresponding to the point M on S, the tangent at Π to one of these curves passes through the focal points F_1' and F_2' of the lines MM_1 and MM_2 respectively, and the tangent to the other curve passes through the other focal points F_1'' and F_2''. Since the set of points M, M_1, F_1', F_1'' are harmonic and likewise the points M, M_2, F_2', F_2'', it follows that the line joining M_1 and M_2 passes through Π. As the surfaces S and S' bear to S_1 a relation similar to that of S_1 and S_2 to S, it follows that the point Π lies also on the line MM'.

If X, Y, Z denote the coördinates of Π, in order that it be the intersection of the lines MM' and $M_1 M_2$, there must exist functions σ and τ such that

$$X = x + \sigma (x' - x) = x_1 + \tau (x_2 - x_1), \tag{41}$$

and similar expressions for Y and Z. If we eliminate x' from equation (37) and from the equation formed by the last two members of (41), we obtain an equation of the form $ax + bx_1 + cx_2 = 0$. Since the same equation is satisfied by the y's and z's, the expressions a, b, and c must be zero. These equations are consistent in consequence of (36) and give the expressions

$$\sigma = \frac{\lambda_1' \theta_1}{\lambda_1' \theta_1 - \lambda_1 \theta_1'}, \qquad \tau = - \frac{\lambda_2 \theta_1}{\lambda_1 \theta_2 - \lambda_2 \theta_1}.$$

When these values are substituted in (41), we obtain

$$X = \frac{\lambda_1' \theta_1 x' - \theta_1' \lambda_1 x}{\lambda_1' \theta_1 - \theta_1' \lambda_1} = \frac{\lambda_1 \theta_2 x_1 - \lambda_2 \theta_1 x_2}{\lambda_1 \theta_2 - \lambda_2 \theta_1}, \tag{42}$$

which are consistent because of (37).

* Cf. Guichard, loc. cit., p. 491.

From (6) and (9) it follows that the coördinates of the focal points, namely x_i', y_i', z_i' and x_i'', y_i'', z_i'' for $i = 1, 2$, are given by expressions of the form

$$x_i' = \frac{\lambda_i x_i - \rho\theta_i x}{\lambda_i - \rho\theta_i}, \qquad x_i'' = \frac{\lambda_i x_i + \rho\theta_i x}{\lambda_i + \rho\theta_i} \qquad (i = 1, 2). \tag{43}$$

Hence the direction-parameters of the lines $F_1' F_2'$, $F_1'' F_2''$ are proportional to expressions of the form

$$\begin{gathered} x_1 \lambda_1 (\lambda_2 - \rho\theta_2) - x_2 \lambda_2 (\lambda_1 - \rho\theta_1) + \rho x (\lambda_1 \theta_2 - \lambda_2 \theta_1), \\ x_1 \lambda_1 (\lambda_2 + \rho\theta_2) - x_2 \lambda_2 (\lambda_1 + \rho\theta_1) - \rho x (\lambda_1 \theta_2 - \lambda_2 \theta_1), \end{gathered} \tag{44}$$

respectively. With the aid of these values it is readily verified that the point whose coördinates are given by (42) is the intersection of the lines $F_1' F_2'$ and $F_1'' F_2''$.

From the second expression (42) for X we obtain by differentiation

$$\begin{aligned} \frac{\partial X}{\partial u} &= [x_1 \lambda_1 (\lambda_2 + \rho\theta_2) - x_2 \lambda_2 (\lambda_1 + \rho\theta_1) + \rho x (\theta_1 \lambda_2 - \theta_2 \lambda_1)] K_1, \\ \frac{\partial X}{\partial v} &= [x_1 \lambda_1 (\lambda_2 - \rho\theta_2) - x_2 \lambda_2 (\lambda_1 - \rho\theta_1) - \rho x (\theta_1 \lambda_2 - \theta_2 \lambda_1)] K_2, \end{aligned} \tag{45}$$

where

$$K_1 = \frac{\theta_2 \frac{\partial \theta_1}{\partial u} - \theta_1 \frac{\partial \theta_2}{\partial u}}{(\theta_1 \lambda_2 - \theta_2 \lambda_1)^2}, \qquad K_2 = \frac{\theta_2 \frac{\partial \theta_1}{\partial v} - \theta_1 \frac{\partial \theta_2}{\partial v}}{(\theta_1 \lambda_2 - \theta_2 \lambda_1)^2}. \tag{46}$$

Comparing equations (44) and (45), we note that the curves $v = \text{const.}$ on the envelope are tangent to the corresponding lines $F_1'' F_2''$ and the curves $u = \text{const.}$ to the lines $F_1' F_2'$, as previously remarked from the general theory. As the congruences of lines $M' M_1$ and $M' M_2$ bear to the surface S' a relation similar to that of the lines MM_1 and MM_2 to S, the focal points on the lines $M' M_1$ and $M' M_2$ lie on the tangents to the curves $v = \text{const.}$ or $u = \text{const.}$ It follows that the two tangents at a point Π are harmonic with respect to the lines MM' and $M_1 M_2$ through the point.

In order to find the equation satisfied by the coördinates of Π, we differentiate the first of equations (45) with respect to v. After direct reductions we find that

$$\begin{aligned} \frac{\partial^2 X}{\partial u\, \partial v} &+ \frac{K_2}{K_1} \left[\frac{\partial \log \sqrt{\rho}}{\partial u} + \frac{\lambda_2 \frac{\partial \theta_1}{\partial u} - \lambda_1 \frac{\partial \theta_2}{\partial u}}{\theta_1 \lambda_2 - \theta_2 \lambda_1} \right] \frac{\partial X}{\partial u} \\ &+ \frac{K_1}{K_2} \left[\frac{\partial \log \sqrt{\rho}}{\partial v} + \frac{\lambda_2 \frac{\partial \theta_1}{\partial v} - \lambda_1 \frac{\partial \theta_2}{\partial v}}{\theta_1 \lambda_2 - \theta_2 \lambda_1} \right] \frac{\partial X}{\partial v} = 0. \end{aligned} \tag{47}$$

It is evident that ordinarily the invariants of this equation are not equal.

We inquire whether the point Π ever coincides with one of the suite defined by equation (39) for all values of u and v. If we equate this value of x'_c to the second expression (42) for X, we are led to the condition $\theta'_1 \lambda_1 = -c$, which by (27) necessitates $\theta_2/\theta_1 = \text{const.}$ In this case S_1 and S_2 are homothetic.

Gathering together the above results we have

THEOREM 6. *If S, S_1, S_2, S' are four surfaces of a quatern, the plane Π of four corresponding points M, M_1, M_2, M' touches its envelope in the intersection Π of the lines MM' and $M_1 M_2$; the parametric lines on the envelope form a conjugate system (with invariants ordinarily unequal) whose tangents are harmonic to the lines MM' and $M_1 M_2$ and contain the focal points of the lines MM_1, MM_2, $M' M_1$, $M' M_2$.*

5. Relations between transformations K and radial transformations

If ω is a solution of equation (1) which is linearly independent of x, y, z, the surface $\bar{S}$ whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ are given by

$$\bar{x} = x/\omega, \qquad \bar{y} = y/\omega, \qquad \bar{z} = z/\omega \tag{48}$$

is referred to a conjugate system with equal point invariants. In fact, it is easily shown that the coördinates of $\bar{S}$ satisfy the equation

$$\frac{\partial^2 \bar{\theta}}{\partial u\, \partial v} + \frac{\partial}{\partial v} \log \sqrt{\rho}\, \omega \frac{\partial \bar{\theta}}{\partial u} + \frac{\partial}{\partial u} \log \sqrt{\rho}\, \omega \frac{\partial \bar{\theta}}{\partial v} = 0. \tag{49}$$

We shall say that $\bar{S}$ is obtained from S by a *radial transformation.*

If θ is a function linearly independent of x, y, z, and ω, we shall find that the surfaces S_1 and $\bar{S}_1$ resulting from S and $\bar{S}$ by transformations K, determined by θ and $\bar{\theta}$ respectively, where

$$\bar{\theta} = \theta/\omega, \tag{50}$$

are in the relation of a radial transformation. In fact, we shall show that the coördinates x_1, y_1, z_1 and $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ of S_1 and $\bar{S}_1$ respectively are in the relations

$$\bar{x}_1 = x_1/\omega_1, \qquad \bar{y}_1 = y_1/\omega_1, \qquad \bar{z}_1 = z_1/\omega_1, \tag{51}$$

where ω_1 is given by the quadratures

$$\frac{\partial \omega_1}{\partial u} = \frac{\rho}{\lambda}\left[(\omega_1 - \omega)\frac{\partial \theta}{\partial u} + \theta \frac{\partial \omega}{\partial u}\right], \qquad \frac{\partial \omega_1}{\partial v} = -\frac{\rho}{\lambda}\left[(\omega_1 - \omega)\frac{\partial \theta}{\partial v} + \theta \frac{\partial \omega}{\partial v}\right]. \tag{52}$$

As equations (52) are of the form of (10), it is evident that they are consistent and that ω_1 is a solution of the point equation of S_1, namely (5).

The equations of the transformation from $\bar{S}$ to $\bar{S}_1$, analogous to (10), are

$$\begin{aligned}\frac{\partial \bar{x}_1}{\partial u} &= \frac{\rho\omega^2}{\bar{\lambda}}\left[\left(\frac{x_1}{\omega_1} - \frac{x}{\omega}\right)\frac{\partial}{\partial u}\left(\frac{\theta}{\omega}\right) + \frac{\theta}{\omega}\frac{\partial}{\partial u}\left(\frac{x}{\omega}\right)\right],\\ \frac{\partial \bar{x}_1}{\partial v} &= -\frac{\rho\omega^2}{\bar{\lambda}}\left[\left(\frac{x_1}{\omega_1} - \frac{x}{\omega}\right)\frac{\partial}{\partial v}\left(\frac{\theta}{\omega}\right) + \frac{\theta}{\omega}\frac{\partial}{\partial v}\left(\frac{x}{\omega}\right)\right].\end{aligned} \tag{53}$$

When the value of $\bar{x}_1$ from (51) is substituted in (53) and in the reduction equations (52) are used, it is found that the necessary and sufficient condition that these equations be satisfied is that

$$\bar{\lambda} = \omega_1 \lambda. \tag{54}$$

With the aid of equations (52) it is readily shown that this value of $\bar{\lambda}$ satisfies the equations

$$\frac{\partial \bar{\lambda}}{\partial u} = -\rho\omega^2 \frac{\partial}{\partial u}\left(\frac{\theta}{\omega}\right), \qquad \frac{\partial \bar{\lambda}}{\partial v} = \rho\omega^2 \frac{\partial}{\partial v}\left(\frac{\theta}{\omega}\right), \tag{55}$$

which are analogous to equations (8). Hence we have

THEOREM 7. *If two surfaces S and $\bar{S}$ are in the relation of a radial transformation and a transformation K of S is known, it is possible to find by quadratures a transformation K of $\bar{S}$ such that the new surfaces S_1 and $\bar{S}_1$ are in the relation of a radial transformation.*

If in particular $\theta = \omega$, it follows from (55) that $\bar{\lambda}$ is constant. When we take $\bar{\lambda} = -1$, equation (54) becomes

$$\omega_1 = -1/\lambda, \tag{56}$$

which we have seen in § 3 is a solution of equation (5). With the aid of equations (8) it can be shown that this value of ω_1 satisfies equations (52). Now $\bar{\theta} = 1$, consequently $\bar{S}_1$ is associate to $\bar{S}$, as shown in § 1. From these results we have

THEOREM 8. *If a surface S is subjected to a radial transformation determined by a function θ, and if the associate of the resulting surface, determined by the parametric conjugate system, undergoes a radial transformation determined by $-\lambda$, where λ is given by the quadratures* (8), *the new surface S_1 and S are in the relation of the transformation K determined by θ.*

Suppose now that we have a quatern of surfaces S, S_1, S_2, S', expressed in terms of the same functions appearing in § 3. If θ_2 determines a radial transformation of S, a comparison of equations (52) and (27) reveals the fact that the function $\omega_1 = -\theta_1'$. On the other hand $\omega_2 = -1/\lambda_2$. Again, since the radial transformation function of S_1 differs only in sign from the function θ_1' of the transformation of S_1 into S', it follows that $\omega' = 1/\lambda_1'$.

Hence the four surfaces $\bar{S}$, $\bar{S}_1$, $\bar{S}_2$, $\bar{S}'$, whose x-coördinates are respectively of the form

$$(57) \qquad \bar{x} = x/\theta_2, \qquad \bar{x}_1 = -x_1/\theta_1', \qquad \bar{x}_2 = -\lambda_2 x_2, \qquad \bar{x}' = \lambda_1' x',$$

are radial transforms of the surfaces of the original quatern; and these surfaces themselves form a quatern of the special type of Theorem 5, the surfaces $\bar{S}$ and $\bar{S}_2$ being associate and likewise $\bar{S}_1$ and $\bar{S}'$. The transformation functions of this new quatern are

$$(58) \qquad \begin{gathered} \bar{\theta}_1 = \theta_1/\theta_2, \qquad \bar{\lambda}_1 = -\lambda_1 \theta_1'; \qquad \bar{\theta}_2 = -\bar{\lambda}_2 = 1; \qquad \bar{\theta}_1' = -\bar{\lambda}_1' = -1; \\ \bar{\theta}_2' = \lambda_1 \theta_1', \qquad \bar{\lambda}_2' = -\theta_1/\theta_2. \end{gathered}$$

It is readily shown that these functions and the coördinates (57) satisfy equations analogous to (37) and (38).

In like manner the four surfaces whose coördinates are of the form

$$x/\theta_1, \qquad -\lambda_1 x_1, \qquad -x_2/\theta_2', \qquad \lambda_2' x'$$

form a quatern in radial relation to the corresponding surfaces of the other two quaterns.

Accordingly we have

THEOREM 9. *When a quatern of surfaces is known, two other quaterns can be found without quadratures, each of which consists of two pairs of associate surfaces.*

6. W-CONGRUENCES ASSOCIATED WITH A TRANSFORMATION K

We consider a surface S and the associate surface S_0 whose coördinates x_0, y_0, z_0 are given by quadratures of the form

$$(59) \qquad \frac{\partial x_0}{\partial u} = -\rho \frac{\partial x}{\partial u}, \qquad \frac{\partial x_0}{\partial v} = \rho \frac{\partial x}{\partial v},$$

as follows from (11). These coördinates are solutions of

$$(60) \qquad \frac{\partial^2 \theta_0}{\partial u\, \partial v} = \frac{\partial \log \sqrt{\rho}}{\partial v} \frac{\partial \theta_0}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u} \frac{\partial \theta_0}{\partial v},$$

as is readily shown, if we note that x is a solution of equation (1). If we put

$$(61) \qquad \sigma = \theta_0/\sqrt{\rho},$$

the function σ is a solution of the equation

$$(62) \qquad \frac{\partial^2 \sigma}{\partial u\, \partial v} = \sqrt{\rho} \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sqrt{\rho}}\right) \sigma .$$

In accordance with the theory of infinitesimal deformation of surfaces the surface Σ, whose coördinates ξ, η, ζ are given by quadratures of the form

$$\text{(63)} \qquad \frac{\partial \xi}{\partial u} = z_0 \frac{\partial y}{\partial u} - y_0 \frac{\partial z}{\partial u}, \qquad \frac{\partial \xi}{\partial v} = z_0 \frac{\partial y}{\partial v} - y_0 \frac{\partial z}{\partial v},$$

corresponds to S with orthogonality of linear elements.* For the sake of brevity we say that Σ is the *ortho-surface* of S determined by S_0.

By means of (59), equations (63) are reducible to

$$\text{(64)} \qquad \frac{\partial \xi}{\partial u} = -\frac{1}{\rho}\left(z_0 \frac{\partial y_0}{\partial u} - y_0 \frac{\partial z_0}{\partial u} \right), \qquad \frac{\partial \xi}{\partial v} = \frac{1}{\rho}\left(z_0 \frac{\partial y_0}{\partial v} - y_0 \frac{\partial z_0}{\partial v} \right).$$

If we put

$$\text{(65)} \qquad x_0 = \sqrt{\rho}\,\alpha, \qquad y_0 = \sqrt{\rho}\,\beta, \qquad z_0 = \sqrt{\rho}\,\gamma,$$

the functions α, β, γ are solutions of equation (62) and equations (64) assume the Lelieuvre form

$$\text{(66)} \qquad \frac{\partial \xi}{\partial u} = \left(\beta \frac{\partial \gamma}{\partial u} - \gamma \frac{\partial \beta}{\partial u} \right), \qquad \frac{\partial \xi}{\partial v} = -\left(\beta \frac{\partial \gamma}{\partial v} - \gamma \frac{\partial \beta}{\partial v} \right).$$

In like manner there is a surface S_{10} associate to a surface S_1 resulting from S by a transformation K and its coördinates, x_{10}, y_{10}, z_{10}, are given by

$$\text{(67)} \qquad \frac{\partial x_{10}}{\partial u} = -\rho_1 \frac{\partial x_1}{\partial u}, \qquad \frac{\partial x_{10}}{\partial v} = \rho_1 \frac{\partial x_1}{\partial v},$$

where, as follows from (29),

$$\text{(68)} \qquad \rho_1 = \lambda^2/\rho\theta^2.$$

It can readily be shown that the four surfaces S, S_0, S_1, S_{10} satisfy the requirements of Theorem 5, so that S_0 and S_{10} are in the relation of a transformation K. In fact, in this case equation (40) becomes

$$\text{(69)} \qquad \theta(x_{10} - x_0) = \lambda(x_1 - x).$$

If the expressions (10) for $\partial x_1/\partial u$ and $\partial x_1/\partial v$ be substituted in (67) and $(x_1 - x)$ be replaced by its value from (69), we have

$$\text{(70)} \qquad \frac{\partial}{\partial u}(\theta x_{10}) = x_0 \frac{\partial \theta}{\partial u} + \frac{\lambda}{\rho} \frac{\partial x_0}{\partial u}, \qquad \frac{\partial}{\partial v}(\theta x_{10}) = x_0 \frac{\partial \theta}{\partial v} - \frac{\lambda}{\rho} \frac{\partial x_0}{\partial v}.$$

In a manner analogous to (65) we put

$$\text{(71)} \qquad x_{10} = \sqrt{\rho_1}\,\alpha_1 = -\frac{\lambda}{\sqrt{\rho}\,\theta}\,\alpha_1.$$

As defined by equations (8), the function λ is a solution of equation (60).

* E., p. 382.

Consequently the function σ_1, defined by

$$(72)\qquad \sigma_1 = \lambda/\sqrt{\rho},$$

is a solution of (62). In consequence of equations (71), (72), and (8), equations (70) can be written in the form

$$(73)\quad \frac{\partial}{\partial u}(\sigma_1\alpha_1) = -\left(\sigma_1\frac{\partial\alpha}{\partial u} - \alpha\frac{\partial\sigma_1}{\partial u}\right),\qquad \frac{\partial}{\partial v}(\sigma_1\alpha_1) = \left(\sigma_1\frac{\partial\alpha}{\partial v} - \alpha\frac{\partial\sigma_1}{\partial v}\right).$$

Moreover, the coördinates ξ_1, η_1, ζ, of Σ_1, the ortho-surface to S_1 determined by S_{10}, are given by the quadratures

$$(74)\qquad \frac{\partial\xi_1}{\partial u} = \left(\beta_1\frac{\partial\gamma_1}{\partial u} - \gamma_1\frac{\partial\beta_1}{\partial u}\right),\qquad \frac{\partial\xi_1}{\partial v} = -\left(\beta_1\frac{\partial\gamma_1}{\partial v} - \gamma_1\frac{\partial\beta_1}{\partial v}\right).$$

From these results we know that Σ and Σ_1 can be so placed in space that they are the focal surfaces of a W-congruence, and that the following relations hold:

$$(75)\qquad \begin{gathered} \xi_1 - \xi = \beta\gamma_1 - \beta_1\gamma,\qquad \eta_1 - \eta = \gamma\alpha_1 - \gamma_1\alpha, \\ \zeta_1 - \zeta = \alpha\beta_1 - \alpha_1\beta.^* \end{gathered}$$

Each solution of equation (62) determines a surface associate to Σ. We denote by Σ_{01} the associate surface corresponding to the solution σ_1 given by (72), and by Σ_{02} the associate surface determined by the solution

$$(76)\qquad \sigma_2 = 1/\sqrt{\rho}.$$

The first of these gives rise to the W-congruence whose focal surfaces are Σ and Σ_1. The surface Σ_{01} determines an ortho-surface to Σ, namely $\bar{S}$, whose coördinates $\bar{x}, \bar{y}, \bar{z}$ are

$$(77)\qquad \bar{x} = \sigma_1\alpha_1,\qquad \bar{y} = \sigma_1\beta_1,\qquad \bar{z} = \sigma_1\gamma_1,$$

where $\alpha_1, \beta_1, \gamma_1$ are given by (71). The surface $\bar{S}_0$ whose coördinates are

$$(78)\qquad \bar{x}_0 = \alpha/\sigma_1,\qquad \bar{y}_0 = \beta/\sigma_1,\qquad \bar{z}_0 = \gamma/\sigma_1,$$

is associate to $\bar{S}$ and is an ortho-surface of Σ_1.†

The equation for Σ_1 analogous to (62), namely

$$(79)\qquad \frac{\partial^2\bar{\sigma}}{\partial u\,\partial v} = \sqrt{\rho_1}\frac{\partial^2}{\partial u\,\partial v}\left(\frac{1}{\sqrt{\rho_1}}\right)\bar{\sigma},$$

admits the solution $1/\sigma_1$. This solution determines an associate surface of Σ_1 which leads to the ortho-surface $\bar{S}_0$ of Σ_1. Moreover, the joins of corresponding points on this associate surface and Σ_{01} are concurrent.‡

* Cf. E., pp. 417–420.

† Cf. E., p. 420.

‡ Cf. Darboux, *Leçons*, vol. 4, p. 69.

We observe that $1/\sqrt{\rho_1}$ also is a solution of equation (79). We denote by Σ_{10} the surface associate to Σ_1 determined by this solution. If we apply the Moutard transformation of the form (73) to $1/\sqrt{\rho}$, we obtain $1/\sqrt{\rho_1}$. In fact, it is readily shown that the equations

$$(80)\qquad \begin{aligned} \frac{\partial}{\partial u}\left(\sigma_1 \frac{1}{\sqrt{\rho_1}}\right) &= -\left(\sigma_1 \frac{\partial}{\partial u}\frac{1}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho}}\frac{\partial \sigma_1}{\partial u}\right), \\ \frac{\partial}{\partial v}\left(\sigma_1 \frac{1}{\sqrt{\rho_1}}\right) &= \left(\sigma_1 \frac{\partial}{\partial v}\frac{1}{\sqrt{\rho}} - \frac{1}{\sqrt{\rho}}\frac{\partial \sigma_1}{\partial v}\right), \end{aligned}$$

reduce to equations (8). Making use of results which we have established elsewhere,* we have

THEOREM 10. *The surfaces Σ_{02} and Σ_{10}, associate to Σ and Σ_1, and determined by $1/\sqrt{\rho}$ and $1/\sqrt{\rho_1}$ respectively, are such that the lines joining corresponding points on these surfaces form a congruence whose developables meet them in the conjugate parametric system, which has equal tangential invariants; and the lines of intersection of the tangent planes to Σ_{02} and Σ_{10} form a congruence whose developables are parametric and whose focal planes are harmonic to these tangent planes.*

We say that two surfaces, related as Σ_{02} and Σ_{10} are in this theorem, are in the relation of a *transformation* Ω.

We turn to the consideration of the W-congruence, determined by the solution $\sigma_2 = 1/\sqrt{\rho}$ of equation (62), of which Σ is one of the focal surfaces. Since S is the corresponding ortho-surface of Σ, the functions α_2, β_2, γ_2 for the second sheet Σ_2 are given by the equations

$$(81)\qquad \alpha_2 = \sqrt{\rho}\, x, \qquad \beta_2 = \sqrt{\rho}\, y, \qquad \gamma_2 = \sqrt{\rho}\, z,$$

which are analogous to equations (77). These functions are solutions of the equation

$$(82)\qquad \frac{\partial^2 \sigma'}{\partial u\, \partial v} = \frac{1}{\sqrt{\rho}} \frac{\partial^2 \sqrt{\rho}}{\partial u\, \partial v} \cdot \sigma',$$

as is readily shown. Moreover, from (59), (65), and (81) it follows that

$$(83)\qquad \begin{aligned} \frac{\partial}{\partial u}\left(\frac{\alpha_2}{\sqrt{\rho}}\right) &= -\left(\frac{1}{\sqrt{\rho}}\frac{\partial \alpha}{\partial u} - \alpha \frac{\partial}{\partial u}\frac{1}{\sqrt{\rho}}\right), \\ \frac{\partial}{\partial v}\left(\frac{\alpha_2}{\sqrt{\rho}}\right) &= \left(\frac{1}{\sqrt{\rho}}\frac{\partial \alpha}{\partial v} - \alpha \frac{\partial}{\partial v}\frac{1}{\sqrt{\rho}}\right), \end{aligned}$$

which are of the same form as (73).

In like manner the function $\bar{\sigma}_1 = 1/\sqrt{\rho_1}$ determines a W-congruence for

* Rendiconti, loc. cit.

which Σ_1 is one of the focal surfaces. Since S_1 is the corresponding orthosurface to Σ_1, the functions α', β', γ' for the second focal sheet Σ' are given by

$$(84) \qquad \alpha' = \sqrt{\rho_1}\, x_1 = -\frac{\lambda}{\sqrt{\rho}\,\theta} x_1, \qquad \beta' = -\frac{\lambda}{\sqrt{\rho}\,\theta} y_1, \qquad \gamma' = -\frac{\lambda}{\sqrt{\rho}\,\theta} z_1.$$

It is easily found that the function

$$(85) \qquad \sigma_2' = \sqrt{\rho}\,\theta$$

is a solution of equation (82). If we substitute in equations (10) the expressions for x_1 and x which follow from equations (81) and (84), the resulting equations may be given the form

$$(86) \qquad \begin{aligned} \frac{\partial}{\partial u}(\sigma_2' \alpha') &= -\left(\sigma_2' \frac{\partial \alpha_2}{\partial u} - \alpha_2 \frac{\partial \sigma_2'}{\partial u}\right), \\ \frac{\partial}{\partial v}(\sigma_2' \alpha') &= \left(\sigma_2' \frac{\partial \alpha_2}{\partial v} - \alpha_2 \frac{\partial \sigma_2'}{\partial v}\right). \end{aligned}$$

Comparing these equations with (73), we observe that the lines joining corresponding points on the surfaces Σ_2 and Σ' form a W-congruence, for which these are the focal surfaces. It remains to find the relations between the coördinates of these surfaces.

From equations (65), (69), (71), (81), and (84), we obtain the relation

$$(87) \qquad \alpha' = \alpha + \frac{\lambda}{\rho\theta}(\alpha_1 - \alpha_2).$$

Analogous to equations (75) are the following:

$$(88) \qquad \begin{gathered} \xi_2 - \xi = \beta\gamma_2 - \gamma\beta_2, \qquad \xi' - \xi_1 = \beta_1 \gamma' - \gamma_1 \beta', \\ \xi' - \xi_2 = \beta_2 \gamma' - \gamma_2 \beta', \end{gathered}$$

which are consistent in view of (87). From these follows also

$$(89) \qquad \xi' = \xi + \frac{\lambda}{\rho\theta}(\beta_2 \gamma_1 - \beta_1 \gamma_2).$$

From the foregoing results we have

THEOREM 11. *When a pair of surfaces S and S_1 in the relation of a transformation K are known, it is possible to find by quadratures four surfaces Σ, Σ_1, Σ_2, Σ' such that the pairs $\Sigma\Sigma_1$, $\Sigma\Sigma_2$, $\Sigma_1\Sigma'$, $\Sigma_2\Sigma'$ are the focal surfaces of four W-congruences and the asymptotic lines on these surfaces correspond.*

We say that four such W-congruences form a *quatern*, which we indicate by $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma')$.

From this theorem we derive

THEOREM 12. *Four surfaces S, S_1, S_2, S' forming a quatern under transformations K determine twelve W-congruences forming six quaterns of W-congruences.*

Let $W(\Sigma, \Sigma_1, \Sigma_3, \Sigma')$ and $W(\Sigma, \Sigma_2, \Sigma_3, \Sigma'')$ be the quaterns determined by S and S_1 and by S and S_2 respectively in accordance with Theorem 11. The pair of surfaces S_1 and S' determine a quatern $W(\Sigma_1, \Sigma_4, \Sigma', \Sigma''')$. But from the nature of the theorem of permutability it follows that there is a quatern $W(\Sigma_2, \Sigma_4, \Sigma'', \Sigma''')$ determined by S_2 and S'. Furthermore, it is readily seen that these focal surfaces are so placed that there exist also the quaterns $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma_4)$ and $W(\Sigma_3, \Sigma', \Sigma'', \Sigma''')$.

7. TRANSFORMATIONS K DETERMINED BY A QUATERN OF W-CONGRUENCES

In this section we shall show that every quatern of W-congruences gives rise to four pairs of surfaces in the relation of transformations K. In doing so we make use of the results of Bianchi* concerning the determination of such quaterns.

Let Σ be a surface referred to its asymptotic lines and defined by equations (66), where α, β, γ are linearly independent solutions of an equation of the form

$$\frac{\partial^2 \sigma}{\partial u\, \partial v} = M\sigma, \tag{90}$$

M being a function of u and v in general. If σ_1 and σ_2 are two independent solutions of this equation, the functions α_1, β_1, γ_1 and α_2, β_2, γ_2, given by quadratures of the form (73), determine two surfaces Σ_1 and Σ_2 whose cartesian coördinates ξ_1, η_1, ζ_1 and ξ_2, η_2, ζ_2 are expressible in the form (75). Moreover, the functions α_i, β_i, γ_i $(i = 1, 2)$ satisfy the equations

$$\frac{\partial^2 \bar{\sigma}_i}{\partial u\, \partial v} = \sigma_i \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_i}\right) \bar{\sigma}_i \qquad (i = 1, 2). \tag{91}$$

In accordance with the theorem of Moutard these equations are satisfied also by the functions σ_1' and σ_2', defined by

$$\frac{\partial}{\partial u}(\sigma_1 \sigma_1') = -\sigma_1^2 \frac{\partial}{\partial u}\left(\frac{\sigma_2}{\sigma_1}\right), \qquad \frac{\partial}{\partial v}(\sigma_1 \sigma_1') = \sigma_1^2 \frac{\partial}{\partial v}\left(\frac{\sigma_2}{\sigma_1}\right), \tag{92}$$

$$\frac{\partial}{\partial u}(\sigma_2 \sigma_2') = -\sigma_2^2 \frac{\partial}{\partial u}\left(\frac{\sigma_1}{\sigma_2}\right), \qquad \frac{\partial}{\partial v}(\sigma_2 \sigma_2') = \sigma_2^2 \frac{\partial}{\partial v}\left(\frac{\sigma_1}{\sigma_2}\right). \tag{93}$$

In the subsequent discussion we assume that the constants of integration are so chosen that

$$\sigma_2 \sigma_2' = -\sigma_1 \sigma_1', \tag{94}$$

* *Lezioni di Geometria Differenziale*, Pisa (1903), vol. 2, pp. 71–74.

in which case we have

$$\sigma_1' \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_1'}\right) = \sigma_2' \frac{\partial^2}{\partial u\, \partial v}\left(\frac{1}{\sigma_2'}\right). \tag{95}$$

Bianchi has shown that the functions α', β', γ', defined by

$$\begin{gathered} \alpha' = \alpha + t(\alpha_1 - \alpha_2), \qquad \beta' = \beta + t(\beta_1 - \beta_2), \\ \gamma' = \gamma + t(\gamma_1 - \gamma_2), \end{gathered} \tag{96}$$

where

$$t = -\frac{\sigma_2}{\sigma_1'} = \frac{\sigma_1}{\sigma_2'}, \tag{97}$$

satisfy equations (86), also the equations

$$\begin{aligned} \frac{\partial}{\partial u}(\sigma_1' \alpha') &= -\left(\sigma_1' \frac{\partial \alpha_1}{\partial u} - \alpha_1 \frac{\partial \sigma_1'}{\partial u}\right), \\ \frac{\partial}{\partial v}(\sigma_1' \alpha') &= \left(\sigma_1' \frac{\partial \alpha_1}{\partial v} - \alpha_1 \frac{\partial \sigma_1'}{\partial v}\right), \end{aligned} \tag{98}$$

and analogous ones for β' and γ', and finally also

$$\frac{\partial^2 \theta'}{\partial u\, \partial v} = M' \theta', \tag{99}$$

where M' is the function equal to either member of (95).

These functions α', β', γ' determine a surface Σ', whose coördinates are given by

$$\begin{gathered} \xi' = \xi + t(\beta_2 \gamma_1 - \beta_1 \gamma_2), \qquad \eta' = \eta + t(\gamma_2 \alpha_1 - \gamma_1 \alpha_2), \\ \zeta' = \zeta + t(\alpha_2 \beta_1 - \alpha_1 \beta_2), \end{gathered} \tag{100}$$

such that Σ_1 and Σ' are the focal surfaces of a W-congruence and likewise Σ_2 and Σ' of another W-congruence. Hence we have a quatern of W-congruences $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma')$.

We suppose that we have a quatern $W(\Sigma, \Sigma_1, \Sigma_2, \Sigma')$ defined in this general way and we introduce three functions ρ, λ, θ, defined by the equations

$$\sigma_1 = \lambda/\sqrt{\rho}, \qquad \sigma_2 = 1/\sqrt{\rho}, \qquad \sigma_2' = \sqrt{\rho}\,\theta. \tag{101}$$

In terms of these functions equation (90) can be given the form (62) and equations (93) reduce to (8). In accordance with (94) we have

$$\sigma_1' = -\theta\sqrt{\rho}/\lambda = 1/\sqrt{\rho_1}, \tag{102}$$

the function $\sqrt{\rho_1}$ being defined by this equation.

In accordance with the general theory of W-congruences and infinitesimal

deformation of surfaces, the surfaces S and S_0 whose coördinates x, y, z and x_0, y_0, z_0 are given by (81) and (65) are ortho-surfaces of Σ and Σ_2 respectively. In like manner the surfaces $\bar{S}$ and $\bar{S}_0$ whose coördinates $\bar{x}$, $\bar{y}$, $\bar{z}$ and $\bar{x}_0$, $\bar{y}_0$, $\bar{z}_0$ are of the form (77) and (78) are ortho-surfaces of Σ and Σ_1; and S_1 and S_{10}, whose coördinates x_1, y_1, z_1 and x_{10}, y_{10}, z_{10} are given by (84) and (71), are ortho-surfaces of Σ_1 and Σ'. Also the quantities $\bar{x}_1$, $\bar{y}_1$, $\bar{z}_1$ and $\bar{x}_{10}$, $\bar{y}_{10}$, $\bar{z}_{10}$ which are of the form

$$\bar{x}_1 = \sqrt{\rho}\,\theta\alpha', \qquad \bar{x}_{10} = \alpha_2/\sqrt{\rho}\,\theta \tag{103}$$

are the coördinates of ortho-surfaces $\bar{S}_1$ and $\bar{S}_{10}$ of Σ_2 and Σ' respectively.

When the values of α_2 and α' given by (81) and (84) are substituted in (86), we obtain equations (4). Hence the surfaces S and S_1 are in the relation of a transformation K. In like manner it can be shown that each pair of surfaces $\bar{S}\bar{S}_1$, $\bar{S}_0\,\bar{S}_{10}$, $S_0\,S_{10}$ are in the relation of a transformation K. Moreover, it follows from Theorem 10 that each of the above four pairs determines a pair of surfaces referred to a conjugate system with equal tangential invariants which are in the relation of a transformation Ω, and the spherical representation of this conjugate system on any one of these surfaces is the same as of the asymptotic lines on one of the surfaces Σ, Σ_1, Σ_2, Σ'. Hence we have

THEOREM 13. *If Σ, Σ_1, Σ_2, Σ' are the focal surfaces of a quatern of W-congruences, the eight associated ortho-surfaces of the former can be arranged into four pairs such that the surfaces of each pair are in the relation of a transformation K; moreover, each pair determines the quatern of W-congruences; also each pair determines two other surfaces which are in the relation of a transformation Ω.*

8. Equations of a transformation K in another form

In applying the transformations K to particular types of surfaces whose transforms are to be of the same type, it is frequently advisable to have the equations of the transformation in another form, which we will now establish.

Let S be a surface referred to a conjugate system with equal point invariants, and let X, Y, Z; X_1, Y_1, Z_1 and X_2, Y_2, Z_2, denote the direction-cosines of the normal to S and of the bisectors of the angles between the coördinate lines. If one of these angles be denoted by 2ω, we have

$$\frac{\partial x}{\partial u} = \sqrt{E}\,(\cos\omega X_1 - \sin\omega X_2), \qquad \frac{\partial x}{\partial v} = \sqrt{G}\,(\cos\omega X_1 + \sin\omega X_2), \tag{104}$$

where E, F, G denote the coefficients of the linear element of S. Since

$F = \sqrt{EG} \cos 2\omega$, we have

$$
(105) \qquad
\begin{aligned}
\frac{\partial X}{\partial u} &= -\frac{D}{\sqrt{E}} \frac{1}{\sin 2\omega} (\sin \omega \, X_1 - \cos \omega \, X_2), \\
\frac{\partial X}{\partial v} &= \frac{D''}{\sqrt{G}} \frac{1}{\sin 2\omega} (\sin \omega \, X_1 + \cos \omega \, X_2),
\end{aligned}
$$

where D and D'' denote the fundamental coefficients of the second order for S.*

If we compare equation (1) with the Gauss equations for S†, we observe that

$$
(106) \qquad \left\{ {12 \atop 1} \right\} = -\frac{\partial \log \sqrt{\rho}}{\partial v}, \qquad \left\{ {12 \atop 2} \right\} = -\frac{\partial \log \sqrt{\rho}}{\partial u},
$$

the Christoffel symbols $\left\{ {rs \atop t} \right\}$ being formed with respect to the linear element of S. From the definition of these symbols‡ it follows that equations (106) lead to the identities

$$
(107) \qquad
\begin{aligned}
\frac{\partial \sqrt{E}}{\partial v} &= -\sqrt{E} \frac{\partial \log \sqrt{\rho}}{\partial v} - \sqrt{G} \cos 2\omega \frac{\partial \log \sqrt{\rho}}{\partial u}, \\
\frac{\partial \sqrt{G}}{\partial u} &= -\sqrt{G} \frac{\partial \log \sqrt{\rho}}{\partial u} - \sqrt{E} \cos 2\omega \frac{\partial \log \sqrt{\rho}}{\partial v}.
\end{aligned}
$$

The expressions for the other Christoffel symbols are as follows:

$$
(108) \qquad
\begin{aligned}
\left\{ {11 \atop 1} \right\} &= \frac{\partial \log \sqrt{E}}{\partial u} + 2 \cot 2\omega \frac{\partial \omega}{\partial u} - \sqrt{\frac{E}{G}} \cos 2\omega \frac{\partial \log \sqrt{\rho}}{\partial v}, \\
\left\{ {11 \atop 2} \right\} &= \frac{E}{G} \frac{\partial \log \sqrt{\rho}}{\partial v} - 2 \sqrt{\frac{E}{G}} \frac{1}{\sin 2\omega} \frac{\partial \omega}{\partial u}, \\
\left\{ {22 \atop 1} \right\} &= \frac{G}{E} \frac{\partial \log \sqrt{\rho}}{\partial u} - 2 \sqrt{\frac{G}{E}} \frac{1}{\sin 2\omega} \frac{\partial \omega}{\partial v}, \\
\left\{ {22 \atop 2} \right\} &= \frac{\partial \log \sqrt{G}}{\partial v} + 2 \cot 2\omega \frac{\partial \omega}{\partial v} - \sqrt{\frac{G}{E}} \cos 2\omega \frac{\partial \log \sqrt{\rho}}{\partial u}.
\end{aligned}
$$

If equations (104) be differentiated and use be made of equation (1) and

* E., p. 116.

† E., p. 154.

‡ E., p. 153.

of the Gauss equations*

$$
(109)\qquad
\begin{aligned}
\frac{\partial^2 x}{\partial u^2} &= \left\{ {11 \atop 1} \right\} \frac{\partial x}{\partial u} + \left\{ {11 \atop 2} \right\} \frac{\partial x}{\partial v} + DX, \\
\frac{\partial^2 x}{\partial v^2} &= \left\{ {22 \atop 1} \right\} \frac{\partial x}{\partial u} + \left\{ {22 \atop 2} \right\} \frac{\partial x}{\partial v} + D'' X,
\end{aligned}
$$

we obtain

$$
(110)\qquad
\begin{aligned}
\frac{\partial X_1}{\partial u} &= \frac{D}{2\sqrt{E}\cos\omega} X + \left(\sqrt{\frac{E}{G}} \frac{\partial \log \sqrt{\rho}}{\partial v} \sin 2\omega - \frac{\partial \omega}{\partial u} \right) X_2, \\
\frac{\partial X_1}{\partial v} &= \frac{D''}{2\sqrt{G}\cos\omega} X - \left(\sqrt{\frac{G}{E}} \frac{\partial \log \sqrt{\rho}}{\partial u} \sin 2\omega - \frac{\partial \omega}{\partial v} \right) X_2, \\
\frac{\partial X_2}{\partial u} &= \frac{-D}{2\sqrt{E}\sin\omega} X - \left(\sqrt{\frac{E}{G}} \frac{\partial \log \sqrt{\rho}}{\partial v} \sin 2\omega - \frac{\partial \omega}{\partial u} \right) X_1, \\
\frac{\partial X_2}{\partial v} &= \frac{D''}{2\sqrt{G}\sin\omega} X + \left(\sqrt{\frac{G}{E}} \frac{\partial \log \sqrt{\rho}}{\partial u} \sin 2\omega - \frac{\partial \omega}{\partial v} \right) X_1.
\end{aligned}
$$

The conditions of integrability of these equations are satisfied in consequence of the Gauss and Codazzi equations† which in the present case can be given the form

$$
(111)\qquad
\begin{aligned}
\frac{DD''}{\sqrt{EG}} \frac{1}{\sin 2\omega} = \frac{\partial}{\partial v} \left(\sqrt{\frac{E}{G}} \sin 2\omega \frac{\partial \log \sqrt{\rho}}{\partial v} \right) & \\
+ \frac{\partial}{\partial u} \left(\sqrt{\frac{G}{E}} \sin 2\omega \frac{\partial \log \sqrt{\rho}}{\partial u} \right)&,
\end{aligned}
$$

and

$$
(112)\qquad
\begin{aligned}
\frac{\partial D}{\partial v} + \frac{\partial \log \sqrt{\rho}}{\partial v} D + \left(\frac{E}{G} \frac{\partial \log \sqrt{\rho}}{\partial v} - 2 \sqrt{\frac{E}{G}} \frac{1}{\sin 2\omega} \frac{\partial \omega}{\partial u} \right) D'' &= 0, \\
\frac{\partial D''}{\partial u} + \frac{\partial \log \sqrt{\rho}}{\partial u} D'' + \left(\frac{G}{E} \frac{\partial \log \sqrt{\rho}}{\partial u} - 2 \sqrt{\frac{G}{E}} \frac{1}{\sin 2\omega} \frac{\partial \omega}{\partial v} \right) D &= 0.
\end{aligned}
$$

We introduce three functions a, b, w by means of equations of the form

$$
(113)\qquad x_1 - x = \frac{1}{\lambda} (aX_1 + bX_2 + wX),
$$

where x_1, y_1, z_1 are the cartesian coördinates of a surface S_1 in the relation of a transformation K with S and λ is the corresponding function defined by

* E., p. 154.

† E., p. 155.

equations (8). In consequence of (113), the fundamental equations (10) can be written

$$
(114)\qquad
\begin{aligned}
\frac{\partial x_1}{\partial u} &= \frac{\rho}{\lambda^2}\left[\left(a\frac{\partial\theta}{\partial u} + \theta\lambda\cos\omega\sqrt{E}\right)X_1\right.\\
&\qquad\left. + \left(b\frac{\partial\theta}{\partial u} - \theta\lambda\sin\omega\sqrt{E}\right)X_2 + w\frac{\partial\theta}{\partial u}X\right],\\
\frac{\partial x_1}{\partial v} &= -\frac{\rho}{\lambda^2}\left[\left(a\frac{\partial\theta}{\partial v} + \theta\lambda\cos\omega\sqrt{G}\right)X_1\right.\\
&\qquad\left. + \left(b\frac{\partial\theta}{\partial v} + \theta\lambda\sin\omega\sqrt{G}\right)X_2 + w\frac{\partial\theta}{\partial v}X\right].
\end{aligned}
$$

When the expression (113) for x_1 is substituted in these equations, we obtain two equations of the form

$$
A_1X_1 + B_1X_2 + C_1X = 0, \qquad A_2X_1 + B_2X_2 + C_2X = 0,
$$

where $A_1, B_1, \cdots, C_2$ are determinate functions which must be zero, since these equations are satisfied also by the Y's and Z's. This gives the following six equations of condition:

$$
(115)\qquad
\begin{aligned}
\frac{\partial a}{\partial u} &= (\rho\theta - \lambda)\sqrt{E}\cos\omega + b\left(\sqrt{\frac{E}{G}}\frac{\partial\log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right)\\
&\qquad + \frac{wD}{2\sqrt{E}\cos\omega},\\
\frac{\partial a}{\partial v} &= -(\rho\theta + \lambda)\sqrt{G}\cos\omega - b\left(\sqrt{\frac{G}{E}}\frac{\partial\log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right)\\
&\qquad + \frac{wD''}{2\sqrt{G}\cos\omega},\\
\frac{\partial b}{\partial u} &= -(\rho\theta - \lambda)\sqrt{E}\sin\omega - a\left(\sqrt{\frac{E}{G}}\frac{\partial\log\sqrt{\rho}}{\partial v}\sin 2\omega - \frac{\partial\omega}{\partial u}\right)\\
&\qquad - \frac{wD}{2\sqrt{E}\sin\omega},\\
\frac{\partial b}{\partial v} &= -(\rho\theta + \lambda)\sqrt{G}\sin\omega + a\left(\sqrt{\frac{G}{E}}\frac{\partial\log\sqrt{\rho}}{\partial u}\sin 2\omega - \frac{\partial\omega}{\partial v}\right)\\
&\qquad + \frac{wD''}{2\sqrt{G}\sin\omega},\\
\frac{\partial w}{\partial u} &= -\frac{D}{2\sqrt{E}}\left(\frac{a}{\cos\omega} - \frac{b}{\sin\omega}\right), \qquad \frac{\partial w}{\partial v} = -\frac{D''}{2\sqrt{G}}\left(\frac{a}{\cos\omega} + \frac{b}{\sin\omega}\right).
\end{aligned}
$$

With the aid of equations (8), (107), (111), and (112) it can be shown that the conditions of integrability of these equations are satisfied.

If we put for the sake of brevity

$$T^2 = a^2 + b^2 + w^2, \tag{116}$$

it follows from (115) that

$$\begin{aligned} T\frac{\partial T}{\partial u} &= (\rho\theta - \lambda)\sqrt{E}\,(a \cos \omega - b \sin \omega), \\ T\frac{\partial T}{\partial v} &= -(\rho\theta + \lambda)\sqrt{G}\,(a \cos \omega + b \sin \omega). \end{aligned} \tag{117}$$

If X', Y', Z' denote the direction-cosines of the normal to S_1, we have from (114),

$$\begin{aligned} \Delta X' = &- w \sin \omega\left(\sqrt{E}\frac{\partial \theta}{\partial v} + \sqrt{G}\frac{\partial \theta}{\partial u}\right)X_1 \\ &+ w \cos \omega\left(-\sqrt{E}\frac{\partial \theta}{\partial v} + \sqrt{G}\frac{\partial \theta}{\partial u}\right)X_2 \\ &+ X\left[a \sin \omega\left(\sqrt{E}\frac{\partial \theta}{\partial v} + \sqrt{G}\frac{\partial \theta}{\partial u}\right)\right. \\ &\left.+ b \cos \omega\left(\sqrt{E}\frac{\partial \theta}{\partial v} - \sqrt{G}\frac{\partial \theta}{\partial u}\right) + \theta\lambda\sqrt{EG}\sin 2\omega\right], \end{aligned} \tag{118}$$

where Δ^2 is equal to the sum of the squares of the coefficients of X_1, X_2, and X.

9. When S and S_1 envelop a two-parameter family of spheres. Transformations D_m of an isothermic surface

As an example of the foregoing method we consider the case where S and S_1 constitute the envelope of a two parameter family of spheres. In this case we must have

$$\begin{gathered} x_1 - x = \frac{R\Delta}{\lambda}(X' - X), \qquad y_1 - y = \frac{R\Delta}{\lambda}(Y' - Y), \\ z_1 - z = \frac{R\Delta}{\lambda}(Z' - Z), \end{gathered} \tag{119}$$

where $R\Delta/\lambda$ is the radius of the spheres.

When the expressions for x_1 and X' given by (113) and (118) are substituted in the first of equations (119), and the coefficients of X_1, X_2, and X on

the two sides of the resulting identity are equated, we get

$$
(120)\qquad
\begin{aligned}
a &= -w \sin \omega\, R\left(\sqrt{E}\frac{\partial \theta}{\partial v} + \sqrt{G}\frac{\partial \theta}{\partial u}\right),\\
b &= -w \cos \omega\, R\left(\sqrt{E}\frac{\partial \theta}{\partial v} - \sqrt{G}\frac{\partial \theta}{\partial u}\right),\\
w &= \Bigg[a \sin \omega \left(\sqrt{E}\frac{\partial \theta}{\partial v} + \sqrt{G}\frac{\partial \theta}{\partial u}\right) + b \cos \omega \left(\sqrt{E}\frac{\partial \theta}{\partial v} - \sqrt{G}\frac{\partial \theta}{\partial u}\right)\\
&\qquad + \theta\lambda \sqrt{EG} \sin 2\omega - \Delta\Bigg] R.
\end{aligned}
$$

By means of the first two of these equations the last is equivalent to

$$
(121)\qquad a^2 + b^2 + w^2 = Rw\,(\theta\lambda\sqrt{EG}\sin 2\omega - \Delta).
$$

In consequence of the first two of equations (120) equation (118) can be written

$$
(122)\qquad \Delta X' = \frac{a}{R}X_1 + \frac{b}{R}X_2 - X\left[\frac{a^2+b^2}{Rw} - \theta\lambda\sqrt{EG}\sin 2\omega\right],
$$

whence it follows that

$$
(123)\qquad \Delta^2 = \frac{a^2}{R^2} + \frac{b^2}{R^2} + \left(\frac{a^2+b^2}{Rw} - \theta\lambda\sqrt{EG}\sin 2\omega\right)^2.
$$

Eliminating Δ from (121) and (123), we obtain

$$
(124)\qquad a^2 + b^2 + w^2 = 2w\theta\lambda\sqrt{EG}\sin 2w\, R,
$$

and consequently from (121) and (124) we have

$$
(125)\qquad \Delta = -\theta\lambda\sqrt{EG}\sin 2\omega.
$$

By means of (120), (121), (125) we obtain the following equations:

$$
(126)\qquad
\begin{aligned}
\sqrt{E}\,(a\cos\omega - b\sin\omega) &= -w\sin 2\omega\sqrt{EG}\frac{\partial\theta}{\partial u}R = -\frac{T^2}{2\theta\lambda}\frac{\partial\theta}{\partial u},\\
\sqrt{G}\,(a\cos\omega + b\sin\omega) &= -w\sin 2\omega\sqrt{EG}\frac{\partial\theta}{\partial v}R = -\frac{T^2}{2\theta\lambda}\frac{\partial\theta}{\partial v}.
\end{aligned}
$$

With the aid of these expressions and (8) equations (117) are reducible to

$$
\begin{aligned}
\frac{\partial \log T^2}{\partial u} &= \left(\frac{1}{\theta} - \frac{\rho}{\lambda}\right)\frac{\partial\theta}{\partial u} = \frac{\partial}{\partial u}\log(\lambda\theta),\\
\frac{\partial \log T^2}{\partial v} &= \left(\frac{1}{\theta} + \frac{\rho}{\lambda}\right)\frac{\partial\theta}{\partial v} = \frac{\partial}{\partial v}\log(\lambda\theta).
\end{aligned}
$$

Hence on integration we have

$$T^2 = 2c\lambda\theta, \tag{127}$$

where c denotes an arbitrary constant.

Substituting in (126) the value (127) of T^2, we get

$$\begin{aligned} \sqrt{E}\,(a \cos \omega - b \sin \omega) &= -c\frac{\partial \theta}{\partial u}, \\ \sqrt{G}\,(a \cos \omega + b \sin \omega) &= -c\frac{\partial \theta}{\partial v}. \end{aligned} \tag{128}$$

If these equations be differentiated with respect to v and u respectively, and in the reduction use be made of the preceding equations, we find

$$(\lambda + \rho\theta)\,\sqrt{G} \cos 2\omega = 0, \qquad (\lambda - \rho\theta)\,\sqrt{E} \cos 2\omega = 0.$$

Hence we must have $2\omega = \pi/2$, that is, the parametric curves are the lines of curvature. From (107) it follows that

$$\sqrt{E} = V/\sqrt{\rho}, \qquad \sqrt{G} = U/\sqrt{\rho}, \tag{129}$$

where U and V are functions of u and v alone respectively, and consequently S is an isothermic surface. As the relation between S and S_1 is reciprocal, it follows that S_1 also is isothermic. We shall show that there exist pairs of isothermic surfaces satisfying these equations.

Suppose that S is an isothermic surface. The parameters of its lines of curvature can be chosen so that

$$\sqrt{E} = \sqrt{G} = 1/\sqrt{\rho} = e^{\phi}, \tag{130}$$

where ϕ is a function thus defined. Now in the above equations

$$\sin \omega = \cos \omega = 1/\sqrt{2}, \qquad D = e^{2\phi}/\rho_1, \qquad D'' = e^{2\phi}/\rho_2,$$

where ρ_1 and ρ_2 are the principal radii of curvature of S. If we put

$$\alpha = \frac{a-b}{\sqrt{2}}, \qquad \beta = \frac{a+b}{\sqrt{2}}, \qquad c = -\frac{1}{m}, \tag{131}$$

and replace θ and λ by $m\theta$ and $m\lambda$, which evidently is consistent with equations (8), equations (115), (128), and (8) are equivalent to

$$\frac{\partial \alpha}{\partial u} = -m\lambda e^{\phi} + me^{-\phi}\,\theta - \frac{\partial \phi}{\partial v}\beta + \frac{we^{\phi}}{\rho_1}, \qquad \frac{\partial \alpha}{\partial v} = \beta\frac{\partial \phi}{\partial u},$$

$$
(132) \quad \begin{aligned}
&\frac{\partial \beta}{\partial u} = \alpha \frac{\partial \phi}{\partial v}, \qquad \frac{\partial \beta}{\partial v} = -\, m\lambda e^{\phi} - m e^{-\phi}\, \theta - \frac{\partial \phi}{\partial u} \alpha + \frac{w e^{\phi}}{\rho_2}, \\
&\frac{\partial w}{\partial u} = -\, \alpha \frac{e^{\phi}}{\rho_1}, \qquad \frac{\partial w}{\partial v} = -\, \beta \frac{e^{\phi}}{\rho_2}, \\
&\frac{\partial \theta}{\partial u} = e^{\phi}\, \alpha, \qquad \frac{\partial \theta}{\partial v} = e^{\phi}\, \beta, \qquad \frac{\partial \lambda}{\partial u} = -\, e^{-\phi}\, \alpha, \qquad \frac{\partial \lambda}{\partial v} = e^{-\phi}\, \beta.
\end{aligned}
$$

Furthermore equation (127) becomes

$$
(133) \qquad \alpha^2 + \beta^2 + w^2 + 2\lambda\theta m = 0.
$$

It is readily shown that the first derivatives of the left hand member of equation (133) are zero, and consequently by choosing the initial values properly we can find solutions of equations (132) satisfying (133). It should be mentioned that all the conditions of integrability of (132) are easily shown to be satisfied.

If in the subsequent discussion X_1, Y_1, Z_1 and X_2, Y_2, Z_2 denote the direction-cosines of the tangents to the curves $v =$ const. and $u =$ const. respectively, it is necessary and sufficient to replace X_1 and X_2 in the foregoing formulas by $(X_1 + X_2)/\sqrt{2}$ and $(-X_1 + X_2)/\sqrt{2}$ respectively, as is evident from (104).

With these changes equations (113) and (114) become

$$
(134) \qquad x_1 - x = \frac{1}{\lambda m}(\alpha X_1 + \beta X_2 + w X)
$$

and

$$
(135) \quad \begin{aligned}
\frac{\partial x_1}{\partial u} &= \frac{e^{-\phi}}{\lambda^2 m}[(\alpha^2 + \theta\lambda m) X_1 + \alpha\beta X_2 + \alpha w X], \\
\frac{\partial x_1}{\partial v} &= -\frac{e^{-\phi}}{\lambda^2 m}[\alpha\beta X_1 + (\beta^2 + \theta\lambda m) X_2 + \beta w X].
\end{aligned}
$$

From these equations we have for the fundamental coefficients E_1, F_1, G_1, of the surface S_1 the expressions

$$
(136) \qquad E_1 = G_1 = \frac{e^{-2\phi}\,\theta^2}{\lambda^2}, \qquad F_1 = 0,
$$

which proves that S_1 also is isothermic.

For this case the direction-cosines X', Y', Z' of the normal to S_1, as given for the general case by (118), are of the form

$$
(137) \qquad X' = \frac{1}{\theta\lambda m}[(w^2 + \theta\lambda m) X + \alpha w X_1 + \beta w X_2].
$$

With the aid of this expression equation (134) may be written

$$(138) \qquad x_1 + \frac{\theta}{w} X' = x + \frac{\theta}{w} X,$$

which shows that θ/w is the radius of the sphere tangent to S and S_1.

However, equations (132)–(138) are equivalent to those found by entirely different processes by Darboux* for the transformation of isothermic surfaces which he discovered and to which Bianchi has given the name *transformations* D_m.

In consequence of the foregoing results we have

THEOREM 14. *If two surfaces* S *and* S_1 *in the relation of a transformation* K *constitute the envelope of a two-parameter family of spheres, the surfaces are isothermic and* S_1 *is obtained from* S *by a transformation* D_m.

In accordance with (136) we define a function ϕ_1 by

$$(139) \qquad e^{\phi_1} = \frac{e^{-\phi}\,\theta}{\lambda}.$$

If X_1', Y_1', Z_1' and X_2', Y_2', Z_2' denote the direction-cosines of the tangents to the parametric curves on S_1, it follows from (135) and (137) that

$$(140) \qquad \begin{aligned} X_1' &= -\frac{1}{m\lambda\theta}[(\alpha^2 + \theta\lambda m) X_1 + \alpha\beta X_2 + \alpha w X], \\ X_2' &= \frac{1}{m\lambda\theta}[\alpha\beta X_1 + (\beta^2 + \theta\lambda m) X_2 + \beta w X]. \end{aligned}$$

On replacing X_1 and X_2 in (105) and (110) by $(X_1 + X_2)/\sqrt{2}$ and $(-X_1 + X_2)/\sqrt{2}$ respectively, we have the first derivatives of X, X_1, and X_2. With the aid of these formulas we obtain from (140)

$$(141) \qquad \begin{aligned} \frac{\partial X'}{\partial u} &= \left[\frac{e^{\phi}}{\rho_1} + \frac{w}{\theta\lambda}(\lambda e^{\phi} - \theta e^{-\phi})\right] X_1', \\ \frac{\partial X'}{\partial v} &= -\left[\frac{e^{\phi}}{\rho_2} + \frac{w}{\theta\lambda}(\lambda e^{\phi} + \theta e^{-\phi})\right] X_2'. \end{aligned}$$

From these expressions and (135) it follows that the principal radii ρ_1' and ρ_2' of S_1 are given by

$$(142) \qquad \begin{aligned} \frac{e^{\phi_1}}{\rho_1'} &= -\left[\frac{e^{\phi}}{\rho_1} + \frac{w}{\theta\lambda}(\lambda e^{\phi} - \theta e^{-\phi})\right], \\ \frac{e^{\phi_1}}{\rho_2'} &= \left[\frac{e^{\phi}}{\rho_2} + \frac{w}{\theta\lambda}(\lambda e^{\phi} + \theta e^{-\phi})\right]. \end{aligned}$$

* Loc. cit., p. 502.

10. Theorem of permutability of transformations D_m

In this section we apply the results of § 3 to an immediate proof of the theorem of permutability of transformations D_m which Bianchi* established at length by direct processes. We are thus able to appreciate the underlying reason for the existence of this theorem.

In the first place we seek solutions α_1', β_1', w_1', θ_1', λ_1' of equations for S_1 analogous to (132) and (133), it being understood that two sets of solutions of the latter equations are known, which we denote by α_i, β_i, w_i, θ_i, λ_i for $i = 1, 2$. The functions θ_1' and λ_1' are given by equations (27) and (36). Hence if α_1' and β_1' are to satisfy

$$\frac{\partial \theta_1'}{\partial u} = e^{\phi_1}\alpha_1' = -\frac{e^{-\phi}\,\theta_1}{\lambda_1}\alpha_1', \qquad \frac{\partial \theta_1'}{\partial v} = -\frac{e^{-\phi}\,\theta_1}{\lambda_1}\beta_1', \tag{143}$$

we must have

$$\begin{gathered} \alpha_1' = -\frac{\alpha_1}{\theta_1}(\theta_1' + \theta_2) + \alpha_2, \qquad \beta_1' = \frac{\beta_1}{\theta_1}(\theta_1' + \theta_2) - \beta_2, \\ \lambda_1' = \frac{\lambda_1}{\theta_1}(\theta_1' + \theta_2) - \lambda_2. \end{gathered} \tag{144}$$

These values of α_1' and β_1' satisfy identically the equations

$$\frac{\partial \alpha_1'}{\partial v} = \beta_1'\frac{\partial \phi_1}{\partial u}, \qquad \frac{\partial \beta_1'}{\partial u} = \alpha_1'\frac{\partial \phi_1}{\partial v}, \tag{145}$$

and when substituted in

$$\begin{aligned} \frac{\partial \alpha_1'}{\partial u} &= -m_1'\lambda_1' e^{\phi_1} + m_1' e^{-\phi_1}\theta_1' - \frac{\partial \phi_1}{\partial v}\beta_1' + w_1'\frac{e^{\phi_1}}{\rho_1'}, \\ \frac{\partial \beta_1'}{\partial v} &= -m_1'\lambda_1' e^{\phi_1} - m_1' e^{-\phi_1}\theta_1' - \frac{\partial \phi_1}{\partial u}\alpha_1' + w_1'\frac{e^{\phi_1}}{\rho_2'}, \end{aligned}$$

and use is made of the foregoing equations, we get

$$\begin{aligned} &A\frac{e^{\phi_1}}{\rho_1'} + (\theta_1 e^{-\phi} - \lambda_1 e^{\phi})B + \lambda_1 e^{\phi}\left(m_2\frac{\lambda_2}{\lambda_1} + m_1'\frac{\theta_2}{\theta_1}\right) \\ &\qquad\qquad - \theta_1 e^{-\phi}\left(m_1'\frac{\lambda_2}{\lambda_1} + m_2\frac{\theta_2}{\theta_1}\right) = 0, \\ &A\frac{e^{\phi_1}}{\rho_2'} + (\theta_1 e^{-\phi} + \lambda_1 e^{\phi})B - \lambda_1 e^{\phi}\left(m_2\frac{\lambda_2}{\lambda_1} + m_1'\frac{\theta_2}{\theta_1}\right) \\ &\qquad\qquad - \theta_1 e^{-\phi}\left(m_1'\frac{\lambda_2}{\lambda_1} + m_2\frac{\theta_2}{\theta_1}\right) = 0, \end{aligned}$$

* Loc. cit., pp. 109–125.

where

$$
(146) \qquad \begin{aligned} A &= w_1' - \frac{w_1}{\theta_1}(\theta_1' + \theta_2) + w_2, \\ B &= \frac{1}{\theta_1}\left[(\theta_1' + \theta_2)(m_1' - m_1) - \frac{1}{\lambda_1}(\alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2)\right]. \end{aligned}
$$

If we put $m_1' = m_2$, these equations become

$$
(147) \qquad A\frac{e^{\phi_1}}{\rho_1'} + \frac{C}{\theta_1}(\theta_1 e^{-\phi} - \lambda_1 e^{\phi}) = 0, \qquad A\frac{e^{\phi_1}}{\rho_2'} + \frac{C}{\theta_1}(\theta_1 e^{-\phi} + \lambda_1 e^{\phi}) = 0,
$$

where

$$
(148) \qquad \begin{aligned} C = (\theta_1' + \theta_2)(m_2 - m_1) - \frac{1}{\lambda_1}(\alpha_1 \alpha_2 + \beta_1 \beta_2 + w_1 w_2) \\ - \frac{m_2}{\lambda_1}(\lambda_2 \theta_1 + \lambda_1 \theta_2). \end{aligned}
$$

Equations (147) are satisfied when $A = C = 0$. From these equations and (144), we have

$$
(149) \qquad \begin{aligned} -\alpha_1' &= \kappa\left[\frac{\Phi_1 \alpha_1}{\theta_1 \lambda_1} + (m_1 - m_2)\alpha_2\right], \\ \beta_1' &= \kappa\left[\frac{\Phi_1 \beta_1}{\theta_1 \lambda_1} + (m_1 - m_2)\beta_2\right], \\ w_1' &= \kappa\left[\frac{\Phi_1 w_1}{\theta_1 \lambda_1} + (m_1 - m_2)w_2\right], \\ \lambda_1' &= \kappa\left[\frac{\Phi_1}{\theta_1} + (m_1 - m_2)\lambda_2\right], \\ \theta_1' &= \kappa\left[\frac{\Phi_1}{\lambda_1} + (m_1 - m_2)\theta_2\right], \qquad \kappa = \frac{1}{m_2 - m_1}, \end{aligned}
$$

where we have put

$$
(150) \qquad \Phi_1 = \alpha_1 \alpha_2 + \beta_1 \beta_2 + \gamma_1 \gamma_2 + m_2(\lambda_2 \theta_1 + \lambda_1 \theta_2).
$$

It is readily shown that these values of θ_1' and w_1' satisfy equations (143) and also

$$
\frac{\partial w_1'}{\partial u} = -\alpha_1' \frac{e^{\phi_1}}{\rho_1'}, \qquad \frac{\partial w_1'}{\partial v} = -\beta_1' \frac{e^{\phi_1}}{\rho_2'},
$$

and consequently the functions (149) satisfy all the conditions necessary to the problem. Hence we have established with Bianchi

THEOREM 15. *If S is an isothermic surface, and two isothermic surfaces S_1 and S_2 are obtained from S by transformations D_{m1} and D_{m2}, there exists a*

transformation D'_{m2} *of* S_1 *into a surface* S' *which can be obtained from* S_2 *also by a transformation* D'_{m1}; *moreover,* S' *can be found without quadratures.*

When S is isothermic, the associate surface defined by equations (11) is the Christoffel transform. Hence, as an immediate consequence of Theorem 5, we have that transformations D_m are commutative with transformations of Christoffel, a result which Bianchi established by direct calculation.

11. Congruences (G) with more than one pair of surfaces in the relation of a transformation K

In this section we consider the consistency of equations (13) and (18). We write the former in the form of the second of equations (15) and express the condition of integrability. This gives

$$\frac{\partial M}{\partial v} = \frac{k}{N^2} k_{-1} + M \frac{\partial \log k}{\partial v} + 2M^2 - \frac{3Mk}{N}, \tag{151}$$

where k_{-1}, given by

$$k_{-1} = 2k - h - \frac{\partial^2 \log k}{\partial u \, \partial v}, \tag{152}$$

is an invariant of the point equation of the second focal surface of tangents to the curves $u = \text{const.}$ on S_0.

The condition of integrability of (151) and of the first of (15) can be reduced to

$$N^2 h \frac{\partial}{\partial v} \log \frac{h}{k} + \left(k_{-1} \frac{\partial k}{\partial u} - k \frac{\partial k_{-1}}{\partial u} \right) + 3N(MN - k)(k_{-1} - h) = 0. \tag{153}$$

Hence the necessary and sufficient condition that (151) and (15) be consistent identically is that

$$k_{-1} = h = (c+1)k, \tag{154}$$

where c denotes an arbitrary constant. In this case it follows from (152) and (154) that

$$\frac{\partial^2 \log k}{\partial u \, \partial v} + 2ck = 0. \tag{155}$$

When $c = 0$, we have

$$h = k = UV, \tag{156}$$

where U and V are arbitrary functions of u and v respectively. When $c \neq 0$, the general integral of (155) is

$$k = \frac{U' V'}{c(1 + UV)^2}, \qquad h = (c+1)k, \tag{157}$$

U and V being arbitrary functions of u and v respectively and a prime indicating differentiation with respect to the argument.

When $c = -1$, $h = 0$ and it is readily shown that in this case S and S_1 coincide. Consequently we exclude this value.

If we take the expressions (156) or (157) for h and k, and determine a and b in accordance with (14′) two arbitrary functions are introduced. Then each of the functions M and N as determined by equations (15), (18), and (151) involves an arbitrary constant, and σ and σ_1 are found by quadratures. In view of these results we have

THEOREM 16. *Whenever the point equation of a surface S_0 has invariants of the form* (156) *or* (157), *there exist on the tangents to the curves $v = const.$ of S_0 an infinity of pairs of points which generate pairs of surfaces in the relations of transformations K; the same is true of the tangents to the curves $u = const.$ of S_0, if $h = k = UV$.*

Returning to the case when equation (153) is not satisfied identically, we differentiate this equation respectively with respect to u and v, and replace the derivatives of M and N by their expressions from (15), (18), and (151). By means of (153) the two resulting equations are reducible to the forms

$$PN^2 + Q = 0, \qquad RN^2 + S = 0, \tag{158}$$

where P, Q, R, S are determinate functions involving h, k, k_{-1} and their derivatives. When we express the conditions that these equations be consistent, and that the expressions for N thus given and of M from (153) shall satisfy equations (15), (18), (151), we obtain the equations for h and k which determine the surfaces S_0. We are not interested in these equations but in the fact that from the form of equations (158) and (16) it follows that there is only one pair of surfaces S and S_1 associated with the given congruence. Hence we have

THEOREM 17. *If a congruence (G) possesses more than one pair of points which generate surfaces in the relation of a transformation K, it possesses an infinity of such pairs.*

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